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ORIGINAL ARTICLE

Generalized mixed quasi trifunction variational inequalities

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Abstract In this paper, we introduce a new class of trifunction variational inequalities, which is called the generalized mixed quasi trifunction variational inequalities. Using the auxiliary principle technique, we suggest and analyze a proximal point method for solving the generalized mixed quasi trifunction variational inequalities. It is shown that the convergence of the proposed method requires only pseudomonotonicity, which is a weaker condition than monotonicity. Our results represent an improvement and refinement of previously known results. Since the generalized mixed quasi trifunction variational inequalities include bifunction variational inequalities and related optimization problems as special cases, results proved in this paper continue to hold for these problems.

1. Introduction

It is well known that the variational inequality theory, which was introduced and considered by Stampacchia (1964), provides us with a unified, innovative and general framework to study a wide class of problems arising in finance, economics, network analysis, transportation, elasticity and optimization, and applied sciences. Variational inequalities have been generalized and extended in several directions using the novel and new techniques. Noor and Oettli (1994) considered and studied a class of variational inequalities involving trifunctions. They discussed the existence and uniqueness of the trifunction variational inequalities using the Fan–Glicksberg–Hoffman Lemma. For the applications and other techniques for solving trifunction variational inequalities (Noor, 2010d; Noor and Noor, 2010a; Noor and Oettli, 1994 and the references therein). We would like to remark that the variational inequalities represent the optimality conditions of the convex functions. For the directionally differentiable convex function, we have the bifunction variational inequalities. We would like to remark that the minimum of the sum of directional differentiable convex function and the nondifferentiable bifunction can be characterized by the mixed quasi bifunction variational inequalities. This has motivated Noor (2010d) to consider

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and analyze a class of bifunction variational inequalities, which is called the multivalued mixed quasi bifunction variational inequality involving the nonlinear term $\varphi(.,.)$. Noor (2010d) has used the auxiliary principle technique to suggest and analyze a proximal point algorithm for solving the multivalued mixed quasi bifunction variational inequalities. It is shown that the convergence of the proximal algorithm requires the pseudomonotonicity of the bifunction and the skew symmetry of the bifunction $\varphi(.,.)$.

Inspired and motivated by the research and activities going on in this fascinating area, we introduce and consider a new class of trifunction variational inequalities, which is called the generalized mixed quasi trifunction variational inequality involving the nonlinear term $\varphi(.,.)$. This class is quite general and unifying one and includes several classes of trifunction, bifunction and classsical variational inequalities as special cases. In recent years, several numerical techniques including projection, resolvent and auxiliary principle have been developed and analyzed for solving variational inequalities. We would like to point out that the projection-type methods and their invariant forms can not be used for solving the trifunction hemivariational inequalities. To overcome this drawback, one usually uses the auxiliary principle technique, which is due to Glowinski et al. (1981). This technique has been used to suggest and analyze several methods for solving trifunction variational inequalities and related optimization problems. It has been shown that a substantial number of numerical methods can be obtained as special cases from this technique (Noor, 1999, 2000, 2004a,b,c, 2006, 2009, 2010a,b,c,d,e; Noor and Noor, 2010a,b; Noor et al., 1993, 2010). In this paper, we again use the auxiliary principle technique to suggest and analyze an implicit method for solving the generalized mixed quasi trifunction variational inequalities. It is shown that the proposed proximal method converges for pseudomonotone operators and the skew-symmetric bifunction $\varphi(.,.)$. Our results can be viewed as a significant extension and generalization of the previously known results for solving classical trifunction and bifunction variational inequalities.

2. Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|.\|$, respectively. Let $C(H)$ be a family of all nonempty compact subset of H. Let $T: H \to C(H)$ be a multivalued operator. Let K be a nonempty closed convex set in H. Let $\varphi(.,.): H \times H \to R \cup \{+\infty\}$ be a continuous bifunction. For a given trifunction $F(.,.,.): K \times K \times K \to C(H)$, we consider the problem of finding $u \in K, v \in T(u)$ such that

$$F(u, v, v-u) + \varphi(v,u) - \varphi(u,u) \geqslant 0, \quad \forall v \in K, \tag{2.1}$$

which is called the *generalized mixed quasi trifunction variational inequalities*.

If T is a single-valued operator, then problem (2.1) is equivalent to finding $u \in K$ such that

$$F(u, T(u), v-u) + \varphi(v,u) - \varphi(u,u) \geqslant 0, \quad \forall v \in K, \tag{2.2}$$

which is called the mixed quasi trifunction variational inequalities. For the applications and numerical results of mixed quasi trifunction variational inequalities (Noor and Noor, 2010a; Noor and Oettli, 1994 and the references therein).

If $F(.,.,.) \equiv B(.,.)$, where $B(.,.)$ is bifunction, then problem (2.1) is equivalent to finding $u \in K, v \in T(u)$ such that

$$B(v, v-u) + \varphi(v,u) - \varphi(u,u) \geqslant 0, \quad \forall v \in K, \tag{2.3}$$

which is known as the generalized (multivalued) mixed quasi bifunction variational inequality, which was introduced and studied by Noor (2010d).

If $F(u, v, v-u) \equiv \langle v, v-u\rangle$, then problem (2.1) is equivalent to finding $u \in K, v \in T(u)$ such that

$$\langle v, v-u\rangle + \varphi(v,u) - \varphi(u,u) \geqslant 0, \quad \forall v \in K, \tag{2.4}$$

which is known as the generalized mixed quasi variational inequality (Carl et al., 2007; Dem'yanov et al., 1996; Giannessi and Maugeri, 1995; Giannessi et al., 2001; Gilbert et al., 2001; Glowinski et al., 1981; Noor, 1975, 1999, 2000, 2004a,b,c, 2006, 2009, 2010a,b,c,d,e; Noor and Noor, 2010a,b; Noor et al., 1993, 2010; Noor and Oettli, 1994; Stampacchia, 1964) for applications and numerical results. In brief, for suitable and appropriate choice of the operator and the spaces, one can obtain several known and new classes of variational inequalities and related optimization problems as special cases of problem (2.1). This shows that problem (2.1) is quite general, flexible and unifying one. Furthermore, It is well-known that a wide class of obstacle, unilateral, contact, free, moving and equilibrium problems arising in mathematical, engineering, economics and finance can be studied in the unified and general framework of problems (2.1)–(2.4) and their special cases, see (Carl et al., 2007; Dem'yanov et al., 1996; Giannessi and Maugeri, 1995; Giannessi et al., 2001; Gilbert et al., 2001; Glowinski et al., 1981; Noor, 1975, 1999, 2000, 2004a,b,c, 2006, 2009, 2010a,b,c,d,e; Noor and Noor, 2010a,b; Noor et al., 1993, 2010; Noor and Oettli, 1994; Stampacchia, 1964).

We also need the following concepts and results.

Lemma 2.1. $\forall u, v \in H$,

$$2\langle u, v\rangle = \|u+v\|^2 - \|u\|^2 - \|v\|^2. \tag{2.5}$$

Definition 2.1. The trifunction $F(.,.,.): K \times K \times K \to H$ and the operator T is said to be *jointly pseudomonotone* with respect to the bifunction $\varphi(.,.)$, iff

$$\begin{aligned} &F(u, v, v-u) + \varphi(v,u) - \varphi(u,u) \geqslant 0 \\ &\quad \Rightarrow -F(v, \mu, u-v) + \varphi(v,u) - \varphi(u,u) \geqslant 0, \\ &\quad \forall u, v \in K, \quad v \in T(u), \mu \in T(v). \end{aligned} \tag{2.6}$$

Definition 2.2. The bifunction $\varphi(.,.)$ is said to be *skew-symmetric*, if,

$$\varphi(u,u) - \varphi(u,v) - \varphi(v,u) + \varphi(v,v) \geqslant 0, \quad \forall u, v \in H.$$

Clearly, if the bifunction $\varphi(.,.)$ is linear in both arguments, then,

$$\varphi(u,u) - \varphi(u,v) - \varphi(v,u) + \varphi(v,v) = \varphi(u-v, u-v) \geqslant 0, \quad \forall u, v \in H,$$

which shows that the bifunction $\varphi(.,.)$ is nonnegative.

Definition 2.3. $\forall u_1, u_2 \in H, w_1 \in T(u_1), w_2 \in T(u_2)$ the operator $T: H \to C(H)$ is said to be *M-Lipschitz continuous*, if there exists a constant $\delta > 0$ such that

$$M(T(u_1), T(u_2)) \leqslant \delta \|u_1 - u_2\|,$$

where $M(.,.)$ is the Hausdorff metric on $C(H)$.

3. Main results

We suggest and analyze a proximal method for generalized mixed quasi trifunction variational inequalities (2.1) using the auxiliary principle technique of Glowinski et al. (1981) as developed by Noor (1999, 2000, 2004a,b,c, 2006, 2009, 2010a,b,c,d,e), Noor and Noor (2010a).

For a given $u \in K$ satisfying (2.1), consider the auxiliary problem of finding a unique $w \in K, \eta \in T(w)$ such that

$$\rho F(w, \eta, v - w) + \langle w - u, v - w\rangle + \varphi(v, w) - \varphi(w, w) \geqslant 0, \quad \forall v \in K, \tag{3.1}$$

where $\rho > 0$ is a constant.

We note that if $w = u$, then clearly w is solution of (2.1). This observation enables us to suggest and analyze the following iterative method for solving (2.1).

Algorithm 3.1. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\rho F(u_{n+1}, \eta_{n+1}, v - u_{n+1}) + \langle u_{n+1} - u_n, v - u_{n+1}\rangle + \varphi(v, u_{n+1}) - \varphi(u_{n+1}, u_{n+1}) \geqslant 0, \quad \forall v \in K, \tag{3.2}$$

$$\eta_n \in T(w_n) : \|\eta_{n+1} - \eta_n\| \leqslant M(T(w_{n+1}), T(w_n)), \tag{3.3}$$

which is known as the proximal method for solving generalized mixed quasi trifunction variational inequalities (2.1).

If $F(u, v, v - u) = B(v, v - u)$, then Algorithm 3.1 reduces to

Algorithm 3.2. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\rho B(\eta_{n+1}, v - u_{n+1}) + \langle u_{n+1} - u_n, v - u_{n+1}\rangle + \varphi(v, u_{n+1}) - \varphi(u_{n+1}, u_{n+1}) \geqslant 0, \quad \forall v \in K,$$
$$\eta_n \in T(w_n) : \|\eta_{n+1} - \eta_n\| \leqslant M(T(w_{n+1}), T(w_n)),$$

for solving the generalized mixed quasi bifunction variational inequality (2.3).

If $F(u, v, v - u) = \langle v, v - u\rangle$, where $T : K \to C(H)$ is a nonlinear multivalued operator, then Algorithm 3.1 reduce to:

Algorithm 3.3. For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho\eta_{n+1} + u_{n+1} - u_n, v - u_{n+1}\rangle + \varphi(v, u_{n+1}) - \varphi(u_{n+1}, u_{n+1}) \geqslant 0, \quad \forall \; v \in K$$
$$\eta_n \in T(w_n) : \|\eta_{n+1} - \eta_n\| \leqslant M(T(w_{n+1}), T(w_n)).$$

Algorithm 3.3 is known as the proximal point algorithm for solving generalized mixed quasi variational inequalities (2.4). In a similar way, one can obtain several iterative methods for equilibrium problems and variational inequalities, see (Noor, 1999, 2000, 2004a,b,c, 2006, 2009, 2010a,b,c,d,e; Noor and Noor, 2010a,b; Noor et al., 1993, 2010; Noor and Oettli, 1994).

We now study the convergence analysis of Algorithm 3.1 using the technique of Noor (2010d). For the sake of completeness and to convey an idea of the technique, we include all the details.

Theorem 3.1. *Let $F(.,.,.)$ and T be jointly pseudomonotone with respect to the bifunction $\varphi(.,.)$ and the bifunction $\varphi(.,.)$ be skew-symmetric. If $u \in K, v \in T(u)$ is a solution of (2.1) and u_{n+1} is an approximate solution obtained from Algorithm 3.1, then*

$$\|u_{n+1} - u\|^2 \leqslant \|u_n - u\|^2 - \|u_{n+1} - u_n\|^2. \tag{3.4}$$

Proof. Let $u \in K, v \in T(u)$ be a solution of (2.1). Then

$$F(u, v, v - u) + \varphi(v, u) - \varphi(u, u) \geqslant 0, \quad \forall v \in K,$$

which implies that

$$-F(v, \mu, u - v) + \varphi(v, u) - \varphi(u, u) \geqslant 0, \quad \forall v \in K, \quad \mu \in T(v), \tag{3.5}$$

since $F(.,.,.)$ and T are jointly pseudomonotone with respect to the bifunction $\varphi(.,.)$.

Taking $v = u_{n+1}$ in (3.5), we have

$$-F(u_{n+1}, \mu_{n+1}, u - u_{n+1}) + \varphi(u_{n+1}, u) - \varphi(u, u) \geqslant 0. \tag{3.6}$$

Now taking $v = u$ in (3.2), we obtain

$$\rho F(u_{n+1}, \eta_{n+1}, u - u_{n+1}) + \langle u_{n+1} - u_n, u - u_{n+1}\rangle + \varphi(u, u_{n+1}) - \varphi(u_{n+1}, u_{n+1}) \geqslant 0. \tag{3.7}$$

From (3.6) and (3.7), we have

$$\begin{aligned}\langle u_{n+1} - u_n, u - u_{n+1}\rangle &\geqslant -\rho F(u_{n+1}, \eta_{n+1}, u - u_{n+1}) + \varphi(u_{n+1}, u_{n+1}) - \varphi(u, u_{n+1}) \\ &\geqslant +\varphi(u_{n+1}, u_{n+1}) - \varphi(u, u_{n+1}) - \varphi(u_{n+1}, u) + \varphi(u, u) \\ &\geqslant 0,\end{aligned} \tag{3.8}$$

since the bifunction $\varphi(.,.)$ is skew-symmetric.

Setting $u = u - u_{n+1}$ and $v = u_{n+1} - u_n$ in (2.5), we obtain

$$2\langle u_{n+1} - u_n, u - u_{n+1}\rangle = \|u - u_n\|^2 - \|u - u_{n+1}\|^2 - \|u_n - u_{n+1}\|^2. \tag{3.9}$$

Combining (3.8) and (3.9), we have

$$\|u_{n+1} - u\|^2 \leqslant \|u_n - u\|^2 - \|u_{n+1} - u_n\|^2,$$

the required result. □

Theorem 3.2. *Let H be a finite dimensional space. If u_{n+1} is the approximate solution obtained from Algorithm 3.1 and $u \in K, v \in T(u)$ is a solution of (2.1), then $\lim_{n\to\infty} u_n = u$.*

Proof. Let $\bar{u} \in K$ be a solution of (2.1). From (3.4), it follows that the sequence $\{\|\bar{u} - u_n\|\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Also from (3.4), we have

$$\sum_{n=0}^{\infty} \|u_{n+1} - u_n\|^2 \leqslant \|u_0 - u\|^2,$$

which implies that

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{3.10}$$

Let $\hat{u}$ be a cluster point of $\{u_n\}$ and the subsequence $\{u_{n_j}\}$ of the sequence $\{u_n\}$ converge to $\hat{u} \in H$. Replacing u_n by u_{n_j} in (3.2) and taking the limit $n_j \to \infty$ and using (3.10), we have

$$F(\hat{u}, \hat{v}, v - \hat{u}) + \varphi(v, \hat{u}) - \varphi(\hat{u}, \hat{u}) \geqslant 0, \quad \forall v \in K,$$

which implies that $\hat{u}$ solves the generalized mixed quasi trifunction variational inequality (2.1) and

$$\|u_{n+1} - u_n\|^2 \leqslant \|u_n - u\|^2.$$

Thus it follows from the above inequality that the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$ and

$$\lim_{n\to\infty} u_n = \hat{u}.$$

It remains to show that $v \in T(u)$. From (3.3) and using the M-Lipschitz continuity of the multivalued operator T, we have

$$||v_n - v|| \leqslant M(T(u_n), T(u)) \leqslant \delta||u_n - u||,$$

which implies that $v_n \to v$ as $n \to \infty$. Now consider

$$\begin{aligned} d(v, T(u)) &\leqslant ||v - v_n|| + d(v, T(u)) \\ &\leqslant ||v - v_n|| + M(T(u_n), T(u)) \\ &\leqslant ||v - v_n|| + \delta||u_n - u|| \quad \to 0 \text{ as } n \to \infty, \end{aligned}$$

where $d(v, T(u)) = inf\{||v - z|| : z \in T(u)\}$. and $\delta > 0$ is the M-Lipschitz continuity constant of the operator T. From the above inequality, it follows that $d(v, T(u)) = 0$. This implies that $v \in T(u)$, since $T(u) \in C(H)$. This completes the proof. □

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